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CRT-2000-14

Centre de recherche sur les transports (C.R.T.)
Centre for Research on Transportation

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April 2000

PUBLICATION
CRT-2000-14

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* *This research was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC).*

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Dépôt légal – Bibliothèque nationale du Québec,
Bibliothèque nationale du Canada, 2000

101-01784/

ABSTRACT

The continuous dynamic network loading problem is one of important problems in the dynamic network equilibrium problems. There are some solution procedures developed for the problem. However, there is so far few theoretical analysis about the continuity of the continuous dynamic network loading operator. In this paper, we explore this issue with a functional analysis approach. We will show that, under some reasonable conditions, the solution of the problem is continuously dependent on its inputs. The result and the corresponding methodology can be used in the future as a concrete basis for further theoretical analysis of the dynamic network equilibrium problem, the convergence of various solution algorithms, the stability analysis for the dynamic network equilibrium problem and the ITS models.

Keywords: Continuous Dynamic Network Loading, Nonlinear Operator, Dynamic Traffic Assignment, ITS.

RÉSUMÉ

Le problème de chargement dynamique continue de réseau est un des problèmes important du problème d'équilibre dynamique sur un réseau. Plusieurs méthodes ont été développées pour résoudre ce problème. Cependant, il n'y a pas beaucoup d'analyses théoriques sur la continuité de l'opérateur de chargement dynamique continue de réseau. Dans cet article, nous explorons cette issue avec une approche d'analyse fonctionnelle. Nous allons montrer que sous certaines conditions raisonnables, la solution du problème est continûment dépendant de ses données. Le résultat et la méthodologie correspondant peuvent être utilisés dans le futur comme une basse concrète pour des analyses théoriques plus exhaustives du problème d'équilibre dynamique, la convergence de plusieurs algorithmes, les analyses de stabilité pour le problème d'équilibre dynamique et de modèle SIT.

Mots-clés: problème de chargement dynamique continue, opérateur non linéaire, affectation dynamique du trafic, Systèmes Intelligents de Transport (SIT).

1 Introduction

The continuous dynamic network loading problem could be briefly described as to find, on a congested network, temporal network performance information: arc traffic volume, arc travel times, and path travel times, for a predetermined set of time-dependent path flow departure distributions in a given time period. Exactly speaking, in the continuous dynamic network loading problem, we are given a road network (N, A) with arc traversal time functions $s(v(t)) = (s_a(v(t)))$ where $v(t)$ is the arc traffic volume vector. For an arbitrary input, that is, a set of path flow departure distributions, we want to know if the path departure distributions can be rationally assigned to the road network, and, if it can, what are the functional forms of its output, including arc traffic volume functions, arc traversal time functions and path travel time functions.

Several mathematical formulations have been suggested by various authors, e.g., Friesz et al(1993), Ran et al(1996), Wu et al(1998), and Xu et al(1999). For all formulations, an arc flow conservation equation, or "continuity equation", which builds the relation among arc traffic volume, the arc entry rate and arc exit rate, was adopted. Here the phrase "continuity equation" is borrowed from Lighthill and Whitham(1955) where they described a macroscopic continuum traffic model. See also Leo and Pretty, 1990. In Friesz et al(1993) and Wu et al(1998), the relation was built in an integration equation

$$v_a(t) = \int_0^t b_a(x)dx - \int_0^t e_a(x)dx,$$

while in Ran et al(1996) and Xu et al(1999) it is expressed in a differential equation

$$v_a'(t) = b_a(t) - e_a(t).$$

In common for the four formulations is that all the formulations assumed the First-In-First-Out (FIFO for short) principle, implicitly or explicitly. In Friesz et al(1993) and Wu et al(1998), the FIFO condition is imposed into the model implicitly. (Notice that when writing down the continuity equation, one has assumed the FIFO condition.) While FIFO condition was explicitly included in the model in Ran et al(1996) in a form of propagation constraints, and in Xu et al(1999) it is expressed in a more straightforward way.

It is noted that Friesz et al (1993) is the first to prove the existence of the FIFO condition for a linear travel time function, while, inspired by their work, Xu et al (1999) prove the existence of FIFO conditions for the whole network and nonlinear travel time function and also provide a finite step solution method, which appears to be the first algorithm for this type of the problem with a rigorous proof.

The continuous dynamic network loading problem may be considered as a subproblem of a dynamic traffic assignment problem. In fact, it is embedded in the dynamic traffic assignment problem. Suppose we are given a set of path departure rate distributions over the studied time period, by solving continuous dynamic network loading problem, which takes the given distributions as inputs, we will get all the performance information of the network related to that input, i.e., temporal network densities, arc travel times and path travel times. The network performance may not be at a good status, measured by either system optimum principle or user optimum (equilibrium) principle. Trying to capture some desired optimum status, one may modify the input depending on the current network performance by some rules.

From both practical and theoretical points of view, we may ask the question: Does a small change in inputs (path departure rate distributions) in this case will result in a small change in outputs (temporal network densities, arc travel times, and path travel times)? In the other words, does the outputs of the continuous dynamic network loading problem continuously depends on its inputs, if the problem has a solution? Obviously, if the answer is negative, then the dynamic traffic equilibrium problem may not possess a solution. On the other hand, a positive answer to the question can contribute in the future to the following areas:

- A concrete basis for further theoretical analysis of the dynamic network equilibrium problem in the framework of the dynamic system. In fact the issue of continuity property has been a long standing question in the literature. The work of this paper has also answered more than affirmatively to the question left by Friesz et al (1993), whether the delay operators are continuous. In a recent paper by Zhu and Marcotte (2000), they have proved that the loading operator is weekly continuous under the condition of strongly FIFO. In this paper, we use the (regular) FIFO condition instead of the strong FIFO condition, and show that the loading operator is Lipschitz continuous instead of the weak continuity, in the norm of the operator space, which is usually referred as strong Lipschitz continuous. However, we note that Zhu and Marcotte (2000)'s work is much beyond establishing the week continuity of the loading operator. As a main contribution, they proved the existence of dynamic user equilibrium based upon Friesz et al (1993)'s infinite variational inequality formulation, although by utilizing the results in this paper the existence can be sustained with regular FIFO assumption which implies the strongly FIFO condition.

- The convergence analysis of the various solution algorithms (Wu et al., 1998) for the dynamic network

equilibrium problem. So far there seems no theoretical paper in the literature which provides proofs of the convergence of the algorithms used to solve the dynamic network equilibrium problems. This type of proofs requires a fundamental study about the properties of dynamic network loading operators.

- The stability of the dynamic user equilibrium, which crucially depends on the Lipschitz continuity of the loading operator. Without the loading operator being justified continuous, one cannot accomplish a smooth loading of the inflows onto the network. If a minor variation of the inflows can result in significant changes of traffic volumes and travel times at certain points, then the commuters may react vehemently enough to destroy any near equilibrium traffic flow patterns. As is well established in the literature (See Cascetta and Cantarella (1991), Friesz et al (1994), Nagurney and Zhang (1997), Smith (1979), Smith (1984), Zhang and Nagurney (1996), Zhang and Nagurney (1997)), stability is fundamental for any user equilibrium to sustain, and that any unstable equilibrium does not prevail. Therefore, from theoretical perspectives, one needs the continuity of the loading operator as one of the foundation blocks to establish the stability of dynamic user equilibrium.

- In some ITS architectures, there is a need for an anticipatory network equilibrium model (See Bottom et al., 1998, Mahmassani and Hawas, 1998), which is supposed to produce a consistency between the anticipated traffic conditions used for the guidance generation and the conditions which actually result from travelers' reactions to the guidance. This is formulated as a fixed point problem, which requires the continuity of the dynamic network loading operator to ensure the existence of the model.

In this paper, we explore the Lipschitz continuity property of the continuous dynamic network loading problem with a functional analysis approach. We shall show that, under some reasonable conditions, the solution of the problem is Lipschitz continuously dependent on its inputs. Since the mathematical analysis is quite demanding, in order to facilitate the reading of the paper, we move the proofs of several lemmas and theorems to the appendices.

The paper is organized as follows. In section 2, we will briefly introduce the formulation for the continuous dynamic network loading problem presented in Xu et al (1999) and discuss the conditions for the existence of the solution. The continuity properties of arc dynamic operators will be discussed in Section 3. In Section 4, we prove the continuity property related to the network dynamic operator. Section 5 concludes the paper.

2 The Mathematical Model and Existence of Solution

In this section we introduce the mathematical formulation of the Continuous Dynamic Network Loading Problem(CDNLP). The conditions which ensure the existence of the solution will also be reviewed. Through out the paper, we denote operators and functions in bold and Italic fonts respectively.

The following is a list of indexes, sets and variables we use in this paper.

N: The set of all nodes of the network;

A: The set of all directed arcs(or ordered node pairs) of the network;

I: The set of all origin/destination pairs, i.e., $I \subseteq N \times N$;

k: A path connecting an origin/destination pair;

K_i: The set of all paths connecting origin/destination pair $i \in I$;

K: The set of all paths, i.e., $K = \cup_{i \in I} K_i$;

K_a: The set of all paths containing arc a, i.e., $K_a = \{k \mid k \in K, a \in k\}$;

(ka): Arc a on path k;

$(ka)_-$: The precedent arc of arc a on path k;

$[0, T]$: The flow departure time interval;

R_+: The studied time period, $R_+ = [0, +\infty)$;

$h_k(t)$: The given flow departure rate distribution of path $k \in K$, $t \in [0,\ T]$;

$b_{ka}(t)$: The flow entry rate of path k into arc a at time t, $k \in K_a$ and $t \in R_+$;

$e_{ka}(t)$: The flow exit rate of path k out of arc a at time t, $k \in K_a$ and $t \in R_+$;

$b_a(t)$: The flow entry rate of arc a at time t, $t \in R_+$,

$$b_a(t) = \sum_{p \in K_a} b_{ka}(t);$$

$e_a(t)$: The flow exit rate of arc a at time t, $t \in R_+$;

$$e_a(t) = \sum_{p \in K_a} e_{ka}(t);$$

$v_a(t)$: The traffic density of arc a at time t, $t \in R_+$;

$s_a(t)$: The traversal time of arc a at time t, $t \in R_+$;

$\tau_a(t)$: The exit time on arc a for flows entering the arc at time t, $t \in R_+$;

$\tau_a^{-1}(t)$: The inverse function of $\tau_a(t)$, if $\tau_a(t)$ is invertible; and

$S_k(t)$: The path travel time of path k at time t, $t \in [0,\ T]$.

The departure time period is restricted to be $[0, T]$ with T a finite real number. The studied time period is set to be R_+. Apparently there is for each $a \in A$ a sufficient large number $T_a > T$ such that $b_a(t) = 0$ when $t > T_a$. Then we say that $b_a(t)$ turns inactive. Therefore there exists a sufficient large number T_s such that when $t > T_s$ all time-dependent variables become zero. Then we say that the system (of arcs) become inactive. Because of the traffic delay, the time at which an arc entry flow function turns into inactive varies from arc to arc. We shall use, in the following, T_a to denote the turning point of $b_a(t)$. And then the turning point for $e_a(t)$ is $\tau_a(T_a)$.

As in Xu et al(1999), we assume that s is positive, non-decreasing and continuously differentiable function over its domain, and h is non-negative, bounded and Lebesgue square integrable on $[0, T]$. (h is measurable and h^2 is integrable.) Mathematically, the continuous dynamic network loading problem may be stated as follows. Given (h, s), find vectors of functions b, v, e and τ, such that the following constraints are satisfied.

(CDNLP)

$$v_a(t) = \begin{cases} \int_0^t b_a(s)ds, & 0 \le t < s_a(0) \\ \int_0^t (b_a(s) - e_a(s))ds, & s_a(0) \le t \le T_a \\ \int_0^{T_a} b_a(s)ds - \int_0^t e_a(s)ds, & T_a < t \le \tau_a(T_a) \\ 0, & t > \tau_a(T_a) \end{cases} \quad \forall a \in A, \tag{1}$$

$$\tau_a(t) = t + s_a(v_a(t)), \forall 0 \le t \le T_a, \forall a \in A, \tag{2}$$

$$\tau'_a(t) = 1 + s'_a(t) > 0, \text{for almost all } 0 \le t \le T_a, \forall a \in A, \tag{3}$$

$$b_{ka}(t) = \begin{cases} h_k(t) & 0 \le t < T, \text{ if } a \text{ is the first arc of path } k, \\ e_{ka}(t) & s_a(0) \le t \le T_a, \end{cases} \tag{4}$$

$$e_a(t) = \sum_{k \in K_a} e_{ka}(t), \tag{5}$$

$$e_a(t) = \begin{cases} 0, & 0 \le t < s_a(0) \\ b_a(\tau_a^{-1}(t))(\tau_a^{-1})'(t), & s_a(0) \le t \le \tau_a(T_a), \\ 0, & t > \tau_a(T_a) \end{cases} \quad \forall a \in A, \tag{6}$$

$$b_a(t) = \sum_{k \in K_a} b_{ka}(t) = \sum_{k \in K_a} e_{ka-}(t), \forall a \in A, \tag{7}$$

$$e_{ka-}(t) = \begin{cases} 0, & 0 \le t < s_a(0) \\ b_{ka-}(\tau_a^{-1}(t))(\tau_a^{-1})'(t), & s_a(0) \le t \le \tau_a(T_a) \quad \forall a \in A, k \in K. \\ 0, & t > \tau_a(T_a) \end{cases} \tag{8}$$

Finally, for a path $k = (a_{k_1}, a_{k_2}, ..., a_{k_{n_k}})$, the path travel time of path k departing at time t, $S_k(t)$ can be calculated by the following recursive equations

$$\begin{cases} t_0 = t; \\ t_i = \tau_{a_{k_i}}(t_{i-1}) = t_{i-1} + s_{a_{k_i}}(t_{i-1}), \ i = 1, 2, ..., n_k; \\ S_k(t) = t_{n_k} - t, \end{cases} \tag{9}$$

for all time $t \in [0, T]$, and for all path $k \in K$. And in an alternative expression, we may write

$$S_k(t) = \sum_{i=1}^{n_k} s_{a_{k_i}}(t_{i-1}), \ \forall t \in [0, T].$$

CDNLP is made up with (1) - (9) with the nonegativity conditions on v, b and e. It is a system of equations of functions, where h and s are given vectors of functions, b, v, e, τ, S are vectors of functions to be determined. It is known that the boundness condition (Xu et al. 1999) ensures the nonegativity conditions.

Theorem 2.1 *If CDNLP has a solution on (N, A), the solution is unique.*

Proof. See Appendix A. □

The next theorem describes important properties of both the arc volume and the arc exit time function.

Theorem 2.2 *If CDNLP has a solution, then v and τ are absolutely continuous, and differentiable almost everywhere on their respective domain.*

Proof. See Xu et al(1999). □

For discussing the existence of a solution, we should first distinct the concept that an arc dynamic problem has a solution from one that a network loading problem has a solution. When for given s_a and b_a, one may find v_a, e_a and τ_a which satisfy (1) - 3 and 6, we say that the arc dynamic problem (b_a, s_a) has a solution.

Evidently, not all cases of the arc dynamic problem have a solution. (It is easy to construct an example in which a given (b_a, s_a) results in $\tau_a'(s_a(0)) < 0$.) But it is also easy to see that when a given (b_a, s_a) results in τ_a which is strictly increasing over its domain, then the arc dynamic problem has a solution. We conclude this observation as the following theorem.

Theorem 2.3 *The necessary and sufficient condition for the arc dynamic problem having a solution is that the derivative of the arc exit time function τ_a' is positive over its domain.*

Proof. Straightforward.

Corollary 2.1 *(Friesz et al, 1993 and Xu et al, 1999) If s_a is linear, then for an arbitrary b_a, the arc dynamic problem has a solution.*

Corollary 2.2 *(Xu et al, 1999) Given an arc dynamic problem (b_a, s_a), if $s_a'(\cdot) < \frac{1}{B_a}$ where B_a is the upper bound of $b_a(t)$, then the arc dynamic problem has a solution.*

Proof. It has been proved that there exists a $\zeta_a > 0$ such that $\tau_a(t) > \zeta_a, \forall t \in [0, T_a]$ under the theorem's condition (Theorem 3.2, Xu et al, 1999). By Theorem 2.3, the arc dynamic problem has a solution. □

For a given (b_a, s_a), we say that the arc dynamic problem (b_a, s_a) satisfies the *flow/arc cooperation conditions*, if

(i) s_a is linear in v_a; or

(ii) $s_a'(\cdot) < \frac{1}{B_a}$ where B_a is the upper bound of b_a.

By that definition we may say that the arc dynamic problem possess a solution when it satisfies the flow/arc cooperation conditions.

Similarly, we describe the following *flow/network cooperation conditions*. Let (N, A) be a road network with an arc traversal time function vector $s = (s_a, a \in A)$, and $h = (h_k, k \in K)$ be a vector of path departure rate distributions. We say that the network dynamic problem satisfies the flow/network cooperation conditions, if for any $a \in A$

(i) $h_k \leq B_k, \forall k \in K$;

(ii) $s_a'(\cdot) < \frac{1}{\sum_{k \in K_a} B_k}$, where B_k is the upper bound of h_k or s' is constant.

We state the following theorem to finish this section, which is a restated Theorem 3.3 of Xu et al (1999).

Theorem 2.4 *Let (N, A) be a road network with an arc traversal time function vector $s = (s_a, a \in A)$, and $h = (h_k, k \in K)$ be a vector of path departure rate distributions. If CDNLD (h, s) satisfies the flow/network cooperation conditions, then CDNLD possesses the unique solution.*

3 Continuity of arc dynamic operators

In this section, we discuss the continuity properties of arc dynamic operators. The arc index will be dropped for convenience for some variables. Throughout this section, we assume that arc entry flow functions are bounded, Lebegue square integrable over the interval $[0, T_a]$, and we denote by $D_a[0, T_a]$, or simply by D_a, the set of feasible arc entry flow functions, i.e.,

$$D_a = \{b_a \in L^2[0, T_a] \mid 0 \leq b_a(t) \leq B_a, \forall t \in [0, T_a]\}.$$

Let E and F be two normed linear spaces and σ be an operator from $D \subseteq E$ into F. The operator σ is said to be continuous at a point $f \in D$, if for a given $\epsilon > 0$, there exists a $\delta > 0$ such that $\| \sigma(f') - \sigma(f) \|_F < \epsilon$ for all $f' \in D$ such that $\| f' - f \|_E < \delta$. Where $\|.\|_E$ and $\|.\|_F$ are corresponding norms on E and F respectively. Further σ is said to be continuous on D if it is continuous at each point of D. An operator σ is said to satisfy the Lipschitz condition on Ω, if there exists a constant L such that $||\sigma(x) - \sigma(y)|| < L||x - y||$ for all x and y in Ω.

Let $\mathbf{v}_a$, and $\mathbf{s}_a$ be operators both from $D_a \subset L^2[0, T_a]$ to C, which are defined as:

$\mathbf{v}_a$: $b_a(t) \in D_a \longrightarrow v_a(t) \in C[0, \tau_a(T_a)]$;

$\mathbf{s}_a$: $b_a(t) \in D_a \longrightarrow \tau_a(t) \in C[0, \tau_a(T_a)]$.

We have the following results.

Theorem 3.1 *Suppose that $\mathbf{s}_a(\mathbf{v})$ is Lipschitz continuous with a Lipschitz constant L_s. Then, operators $\mathbf{v}_a$ and $\mathbf{s}_a$ are Lipschitz continuous on the domain D_a. Namely, there exists $L_v > 0$ and $L_\tau > 0$ such that*

$$||\mathbf{v}_a(b_1) - \mathbf{v}_a(b_2)|| \leq L_v||b_1 - b_2||$$

$$||\mathbf{s}_a(b_1) - \mathbf{s}_a(b_2)|| \leq L_\tau||b_1 - b_2||$$

for any two $b_1, b_2 \in D_a$.

Proof. See Appendix B. □

Theorem 3.2 *Assume that (D_a, s_a) satisfy the flow/arc cooperation conditions. Assume further that there exists $\zeta > 0$ such that $\tau_a'(t) \geq \zeta, \forall t \in [0, T_a]$ for any $b_a \in D_a$. Then operator $\mathbf{v}_a$: $D_a \longrightarrow \mathbf{v}_a(D_a)$ is a one-to-one mapping, and hence is invertible.*

Proof. See Appendix C. □

Let $\mathbf{e_a}$ be an operator from $D_a \subset L^2[0, T_a]$ to $L^2[0, \tau_a(T_a)]$. $\mathbf{e_a}: b_a \longrightarrow e_a$ defined by

$$e_a(t) = b_a(\tau_a^{-1}(t)(\tau_a^{-1})'(t).$$

Next we shall prove that τ' and $\mathbf{e_a(t)}$ is also continuously dependent on the arc entry flow function $b_a(t)$.

Theorem 3.3 *Let D_a and s_a satisfy the flow/arc cooperation conditions. Then τ' and* **e** *continuously depend on $b \in D_a$ with a Lipschitz constant.*

Proof. See Appendix D.

4 Continuity of network dynamic loading operator

In this section, we show the Lipschitz continuity of the network dynamic operator. We are given a road network (N, A) with arc traversal time functions $\{s_a(v_a), a \in A\}$. For a set of path departure rate distributions $\{h_k(t), k \in K\}$, it was proved in Xu et al(1999) that the vehicles will be uniquely assigned to the network, such that the FIFO condition is respected if *flow/network cooperation conditions* are satisfied. Throughout this section, we assume that the flow/network cooperation conditions are satisfied. Our goal of this section is to show that under the flow/network cooperation conditions, the path travel time functions S depend continuously on path departure rate distributions $(h_k(t), k \in P)$. That is, if h has a slight perturbation, then change on S is also slight.

Let

$$D = \{h = (h_k, k \in K) | h_k \in L^2[0, T], 0 \leq h_k(t) \leq B_k, k \in K\}.$$

Define the *network dynamic operator*, denoted by **S**, as the operator which maps $h = (h_k, k \in K) \in D$ to $S = (S_k, k \in K)$, i.e., $\mathbf{S}(h) = S$. It is clear that the network dynamic operator is well defined on its domain D if arc traversal functions satisfy the flow/network cooperation conditions (ii).

Let k be an arbitrary path in K and n_k is the number of arcs of the path which may be expressed as

$$k : a_1 - a_2 - ... - a_{n_k}.$$

Then for any t we have from (9)

$$S_k(t) = s_{a_1}(t) + s_{a_2}(\tau_{a_1}(t)) + ... + s_{a_{n_k}}(\tau_{a_{n_k-1}}(...\tau_{a_1}(t)...)).$$

Because, for any $a \in A$, τ_a is completely determined by b_a and s_a, then S_k is completely determined by $\{b_{a_i}, s_{a_i}, 1 \leq i \leq n_k\}$. Therefore, to prove the continuity property of operator $\mathbf{S}$ in $h \in D$, we need and only need to show that b_a is continuously dependent on h for all $a \in A$.

Let $\alpha = \min_{a \in A}\{s_a(0)\}$ which is a positive number by definition. Let $h_1, h_2 \in D$. Set $T_i = i\alpha$ for $i > 0$. It is obvious that if $f \in L^2[0,T]$ then we have also $f \in L^2[0,T']$ for any $T' \leq T$. We will use notation $f|_{t \leq T'}$ to express f being restricted on domain $[0,T']$, and use $||f||_{t \leq T'}$ to express the $L^2[0,T']$ norm of $f|_{t \leq T'}$.

Theorem 4.1 *Let (N,A) be a road network with arc traversal functions $s_a, a \in A$ and $h = (h_k, k \in K)$ be a set of path departure rate distributions for the network. Assume that h and s satisfy the flow/network cooperation conditions. Then for any arc $a \in A$, the arc flow entry rate b_a depends Lipschitz continuously on h.*

Proof. See Appendix E. □

Now we are on the position to give out the main result of this section.

Theorem 4.2 *Let (N,A) be a road network with arc traversal functions $s_a, a \in A$ and $h = (h_k, k \in P)$ be a set of path departure rate distributions for the network. Assume that h and s satisfy the flow/network cooperation conditions. Then for any path $k \in P$, the path travel time function S_k depends Lipschitz continuously on h.*

5 Conclusion

We have explored in this paper the Lipschitz continuity property of the continuous dynamic network loading problem. We have shown that, under some reasonable conditions, the arc dynamic operators $\mathbf{v}_a, \mathbf{s}_a, \tau_a$ are continuous on their domains. For the network dynamic operator, we have shown that $\mathbf{S}$ continuously maps path departure rate distributions h into path travel time functions $S(h)$. The property can be used in the future for the theoretical analysis of the dynamic network equilibrium problem, the convergence of the various algorithms and the mathematical analysis of the stability of a corresponding dynamical system. It is indicated here that the Lipschitz continuity exists as long as the FIFO conditions or the solution of CDNLP are ensured. It would be interesting to consider if this property still exists for the dynamic network loading problem with

capacity. This type of models is being investigated as well.

6 Appendix A: Proof of Theorem 2.1

Before we prove the theorem, we need following two lemmas.

Lemma 6.1 *Suppose CDNLP has a solution on* (N, A). *Then for any* $t \in R_+$, *if* b_{ka} *is unique on* $[0, t]$ *for all* $a \in A$ *and* $k \in K$, e_{ka} *and* e_a *are unique on* $[s_a(0), \tau_a(t)]$ *for all* $k \in K$, *and* $a \in A$.

Proof. When CDNLP has a solution for a given (h, s), we have $\tau' > 0$ almost everywhere on its domain. With (3), we have τ_a^{-1} is well defined. Since $e_{ka} = b_{ka}(\tau_a^{-1}(t))(\tau_a^{-1})'(t)$ and $e_a = \sum_{k \in K} e_{ka}$, it follows that both e_{ka} and e_a are unique. □

Lemma 6.2 *Suppose CDNLP has a solution on* (N, A). *Then for any* $t \in R_+$, *if* e_{ka} *is unique on* $[s_a(0), t']$ *for all* $k \in K, a \in A$, *then* b_{ka} *and* b_a *are unique on* $[0, t]$.

Proof. It follows from (4). □

Proof of Theorem 2.1: It is clear that we only need to show that b is unique on R_+. Then the uniqueness of v, e and τ follow directly from the formulation.

To show the uniqueness of b, we only need prove that b is unique on $[0, \alpha]$ where $\alpha = \min_{a \in A}\{s_a(0)\}$. Then the uniqueness of b on the whole domain will be ensured by condition that T_a is a finite number and $\tau_a(t) - t > s_a(0), \forall t \in T_a, a \in A$, and by Lemma 6.1 and Lemma 6.2. The uniqueness of b on $[0, s(0)]$ follows the definition of b:

$$b_{ka}(t) = \begin{cases} h_k(t) \\ 0 \end{cases} \quad 0 \le t < \alpha.$$

By using this result and gradually increasing α using Xu et al. (1999), we can conclude the proof. □

7 Appendix B: Proof of Theorem 3.1

Before we prove the theorem, we need four lemmas.

Lemma 7.1 *Suppose the arc dynamic operator has a solution, say,* (b_1, s_1) *and* (b_2, s_2) *have solutions. Let* $b_1, b_2 \in D_a$ *such that* $b_1(t) \le b_2(t)$ *for each* $t \in [0, T_a]$, *then*

$$v_1(t) \le v_2(t) \quad \textit{and} \quad \tau_1(t) \le \tau_2(t)$$

in their respective domains.

Proof. Let v_i and τ_i be the traffic density and exit time of arc a corresponding to b_i for $i = 1, 2$. We have, for $t \in [0, s_a(0)]$

$$v_1(t) = \int_0^t b_1(y)dy \leq \int_0^t b_2(y)dy = v_2(t), \tag{10}$$

and, hence,

$$\tau_1(t) = t + s(v_1(t)) \leq t + s(v_2(t)) = \tau_2(t). \tag{11}$$

Let $t(1) = s_a(0)$ and $t(k+1) = \tau_1(t(k))$, $k \geq 1$.

Assume

$$v_1(t) \leq v_2(t), \quad \forall t \in [0, t(k)] \tag{12}$$

$$\tau_1(t) \leq \tau_1(t), \quad \forall t \in [0, t(k)] \tag{13}$$

Then, with notice to $t(k+1) = \tau_1(t(k)) \leq \tau_2(t(k))$, one has $\tau_1^{-1}(t) \geq \tau_2^{-1}(t), \forall t \in [t(1), t(k+1)]$, and therefore,

$$v_1(t) = \int_{\tau_1^{-1}(t)}^t b_1(y)dy \leq \int_{\tau_2^{-1}(t)}^t b_2(y)dy = v_2(t), \quad \forall t \in [t(1), t(k+1)] \tag{14}$$

Consequently, it follows from that s_a is monotonically nondecreasing that

$$\tau_1(t) = t + s_a(v_1(t)) \leq t + s_a(v_2(t)) = \tau_2(t), \forall t \in [t(1), t(k+1)]. \tag{15}$$

By induction, this concludes that (12) and (13) hold for all $k \geq 1$, with notice to the fact that

$$t(k+1) - t(k) = \tau_1(t(k)) - t(k) = s_a(v_1(t(k)) \geq \ s_a(0) > 0,$$

(12) and (13) are valid for their respective entire domains. □

Lemma 7.2 *Assume that for a given b_a and s_a, the arc dynamic problem possesses a solution. Let τ_a be corresponding arc exit time function. Assume further that there exists a constant $\zeta > 0$ such that $\tau_a'(t) \geq \zeta, \forall t \in [0, T_a]$. Then there exists a constant L_ζ corresponding to ζ such that*

$$|\tau_a^{-1}(t') - \tau_a^{-1}(t'')| \leq L_\zeta |t' - t''|$$

for any $t', t'' \in [s_a(0), \tau_a(T_a)]$, which is equivalent to say that τ_a^{-1} satisfies the Lipschitz condition on its domain.

Proof. Recall the Newton-Lebniz formula for an absolutely continuous function f,

$$f(x') - f(x'') = \int_{x''}^{x'} f'(x)dx.$$

When $|f'(x)| \leq M$ holds over its domain for a positive real number M, f satisfies the Lipschitz condition, particularly,

$$|f(x') - f(x'')| = |\int_{x''}^{x'} f'(x)dx| \leq \int_{x''}^{x'} |f'(x)|dx \leq M|x' - x''|, \quad \forall x', x'', \quad x' \geq x''.$$

Now according to Xu et al(1999),

$$(\tau_a^{-1})'(t) = \frac{1}{\tau_a'(\tau_a^{-1}(t))},$$

which is bounded from upper by $\frac{1}{\zeta}$ under the given condition. Therefore τ_a^{-1} satisfies the Lipschitz continuity condition with Lipschitz constant $L_\zeta = \frac{1}{\zeta}$. □

Let b_1 and b_2 be two arbitrary functions in D_a. Define $\bar{b}$ and $\underline{b}$ as follows.

$$\bar{b}(t) = max\{b_1(t), b_2(t)\}, \forall t \in [0, T_a], \tag{16}$$

$$\underline{b}(t) = \min\{b_1(t), b_2(t)\}, \forall t \in [0, T_a], \tag{17}$$

Clearly the following equalities hold.

$$|b_1(t) - b_2(t)| = \bar{b}(t) - \underline{b}(t), \forall t \in [0, T_a],$$

and

$$||b_1 - b_2|| = ||\bar{b} - \underline{b}||.$$

Lemma 7.3 *Assume that the arc dynamic operator (D_a, s_a) satisfies the flow/arc cooperation conditions. Let b_1 and b_2 be arbitrary functions in D_a. $\bar{b}$ and $\underline{b}$ are as defined in (16)-(17), $\bar{\tau}$ and $\underline{\tau}$ are corresponding arc exit flow functions. If $|\bar{\tau}(t) - \underline{\tau}(t)| < L, t \in [0, T']$ with $T' > 0$, then we have*

$$\underline{\tau}^{-1}(t) - \bar{\tau}^{-1}(t) \leq L_\zeta L, \forall t \in [s_a(0), \underline{\tau}(T')],$$

where L_ζ is the Lipschitz constant in Lemma 7.2.

Proof. For any $t \in [s_a(0), \underline{\tau}(T')]$, we have by Lemma 7.2

$$\begin{aligned} \underline{\tau}^{-1}(t) - \bar{\tau}^{-1}(t) &= \bar{\tau}^{-1}(\bar{\tau}(\underline{\tau}^{-1}(t))) - \bar{\tau}^{-1}(t) \\ &\leq L_\zeta(\bar{\tau}(\underline{\tau}^{-1}(t)) - t) \\ &< L_\zeta L. \end{aligned}$$

□

To demonstrate subsequent properties, we need the following Cauchy Inequality, or Holder Inequality in general(see, for example, Royden,1988).

Lemma 7.4 *(Cauchy Inequality) Let $f \in L^2[a,b]$. Then the following inequality hold.*

$$\int_a^b |f(t)|dt \leq (b-a)^{\frac{1}{2}} (\int_a^b |f(t)|^2 dt)^{\frac{1}{2}} = (b-a)^{\frac{1}{2}} ||f||.$$

Proof of Theorem 3.1: Let b_1 and b_2 be two arbitrary functions in D_a. We first show that there exist two positive constants L_v and L_τ such that

$$|v_1(t) - v_2(t)| \leq \bar{L_v}||b_1 - b_2||, \quad \text{forall} \quad t \in [0, \tau(T_a)] \tag{18}$$

and

$$|\tau_1(t) - \tau_2(t)| \leq \bar{L_\tau}||b_1 - b_2||, \quad \text{forall} \quad t \in [0, T_a]. \tag{19}$$

Let $\bar{v}(t)$ and $\underline{v}(t)$ be the maximum and minimum traffic densities on arc a at time t induced respectively by $\bar{b}$ and $\underline{b}$ as defined in (16) and (17). Let $\bar{\tau}(t)$ and $\underline{\tau}(t)$ be the maximum and minimum exit times at time t corresponding respectively to $\bar{v}(t)$ and $\underline{v}(t)$.

Let $t(0) = 0$, and $t(i+1) = \underline{\tau}(t(i))$, $i = 0, 1, \cdots$. By Lemma7.4, we have

$$|v_1(t) - v_2(t)| \leq \int_0^t |b_1(y) - b_2(y)|dy \leq T_a^{\frac{1}{2}} ||b_1 - b_2|| \tag{20}$$

where the last equality follows from Holder's Inequality and

$$|\tau_1(t) - \tau_2(t)| = |s_a(v_1(t)) - s_a(v_2(t))| \leq L_s|v_1(t) - v_2(t)| \leq L_s T_a^{\frac{1}{2}} ||b_1 - b_2||, \tag{21}$$

for all $t \in [0, t(1)]$.

In addition, we have, by Lemma 7.1,

$$\begin{aligned} \underline{\tau}(t) &\leq \min\{\tau_1(t), \tau_2(t)\} \\ &\leq \max\{\tau_1(t), \tau_2(t)\} \\ &\leq \bar{\tau}(t), \ \forall t \in [0, t(1)]. \end{aligned} \tag{22}$$

According to the arc FIFO condition, it follows from (22) that, for all $t \in [t(1), t(2)]$

$$\begin{aligned} \bar{\tau}^{-1}(t) &\leq \min\{\tau_1^{-1}(t), \tau_2^{-1}(t)\} \\ &\leq \max\{\tau_1^{-1}(t), \tau_2^{-1}(t)\} \\ &\leq \underline{\tau}^{-1}(t) \\ &\leq \underline{\tau}^{-1}(t(2)) \\ &= t(1) \end{aligned} \tag{23}$$

and hence,

$$\tau_1^{-1}(t), \tau_2^{-1}(t) \in [\bar{\tau}^{-1}(t), \underline{\tau}^{-1}(t)] \subset [0, t(1)]. \tag{24}$$

In view of (23), it is clear that

$$\begin{aligned} \int_{\bar{\tau}^{-1}(t)}^{t} \bar{b}(y)dy &\geq \max\{\int_{\tau_1^{-1}(t)}^{t} b_1(y)dy, \int_{\tau_2^{-1}(t)}^{t} b_2(y)dy\} \\ &\geq \min\{\int_{\tau_1^{-1}(t)}^{t} b_1(y)dy, \int_{\tau_2^{-1}(t)}^{t} b_2(y)dy\} \\ &\geq \int_{\underline{\tau}^{-1}(t)}^{t} \underline{b}(y)dy \end{aligned}$$

which results in, for $t \in [t(1), t(2)]$,

$$\begin{aligned} |v_1(t) - v_2(t)| &= |\int_{\tau_1^{-1}(t)}^{t} b_1(y)dy - \int_{\tau_2^{-1}(t)}^{t} b_2(y)dy| \\ &\leq \int_{\bar{\tau}^{-1}(t)}^{t} \bar{b}(y)dy - \int_{\underline{\tau}^{-1}(t)}^{t} \underline{b}(y)dy \\ &= \int_{\bar{\tau}^{-1}(t)}^{\underline{\tau}^{-1}(t)} \bar{b}(y)dy + \int_{\underline{\tau}^{-1}(t)}^{t} (\bar{b}(y) - \underline{b}(y))dy \\ &\leq B_a(\underline{\tau}^{-1}(t) - \bar{\tau}^{-1}(t)) + \int_{0}^{T_a} (\bar{b}(y) - \underline{b}(y))dy \end{aligned} \tag{25}$$

By Lemma 7.2, we have for the first item at the right-hand side of (25),

$$\begin{aligned}
\underline{\tau}^{-1}(t) - \bar{\tau}^{-1}(t) &= \bar{\tau}^{-1}(\bar{\tau}(\underline{\tau}^{-1}(t))) - \bar{\tau}^{-1}(t) \\
&\leq L_\zeta(\bar{\tau}(\underline{\tau}^{-1}(t)) - t) \\
&= L_\zeta(\bar{\tau}(\underline{\tau}^{-1}(t)) - \underline{\tau}(\underline{\tau}^{-1}(t))) \\
&= L_\zeta(s_a(\bar{v}_a(\underline{\tau}^{-1}(t))) - s_a(\underline{v}_a(\underline{\tau}^{-1}(t)))) \\
&\leq L_\zeta L_s|\bar{v}_a(\underline{\tau}^{-1}(t)) - \underline{v}_a(\underline{\tau}^{-1}(t)))| \\
&\leq L_\zeta L_s T_a^{\frac{1}{2}}||b_1 - b_2|| \qquad (26)
\end{aligned}$$

where L_ζ is the Lipschitz constant corresponding for $\zeta > 0$ in Lemma 7.2, L_s is the Lipschitz constant for function s_a, and the last inequality follows from (20) and (24).

Bringing the estimate of (26) into the right-hand side of (25), we obtain,

$$\begin{aligned}
|v_1(t) - v_2(t)| &\leq B_a L_\zeta L_s T_a^{\frac{1}{2}}||b_1 - b_2|| + T_a^{\frac{1}{2}}||b_1 - b_2|| \\
&= T_a^{\frac{1}{2}}(B_a L_\zeta L_s + 1)||b_1 - b_2|| \qquad (27)
\end{aligned}$$

which yields immediately

$$\begin{aligned}
|\tau_1(t) - \tau_2(t)| &= |s_a(v_1(t)) - s_a(v_2(t))| \\
&\leq L_s|v_1(t) - v_2(t)| \\
&\leq T_a^{\frac{1}{2}} L_s(B_a L_\zeta L_s + 1)||b_1 - b_2||, \quad \forall t \in [t(1), t(2)]. \qquad (28)
\end{aligned}$$

In what follows, we will use the induction method to show that

$$|v_1(t) - v_2(t)| \leq T_a^{\frac{1}{2}}||b_1 - b_2|| \sum_{n=0}^{i-1}(B_a L_\zeta L_s)^n, \quad \forall t \in [t(i-1), t(i)] \qquad (29)$$

and

$$|\tau_1(t) - \tau_2(t)| \leq T_a^{\frac{1}{2}} L_\zeta||b_1 - b_2|| \sum_{n=0}^{i-1}(B_a L_\zeta L_s)^n, \quad \forall t \in [t(i-1), t(i)]. \qquad (30)$$

As in (27) and (28), inequalities (29) and (30) hold for i=1 and 2. We now suppose that inequalities (29) and (30) hold for all $i \leq l$. Let $t(l+1) = \underline{\tau}(t(l))$. Lemma 7.1 suggests that, $\forall t \in [t(l-1), t(l)]$,

$$\begin{aligned}
\underline{\tau}(t) &\leq \min\{\tau_1(t), \tau_2(t)\} \\
&\leq \max\{\tau_1(t), \tau_2(t)\} \leq \bar{\tau}(t),
\end{aligned}$$

Consequently, in light of arc FIFO, we have

$$\begin{aligned}
\bar{\tau}^{-1}(t) &\leq \min\{\tau_1^{-1}(t), \tau_2^{-1}(t)\} \\
&\leq \max\{\tau_1^{-1}(t), \tau_2^{-1}(t)\} \\
\underline{\tau}^{-1}(t) &\leq \underline{\tau}^{-1}(t(l+1)) \\
&= t(l)
\end{aligned} \tag{31}$$

It follows that, for $t \in [t(l), t(l+1)]$, we obtain

$$\begin{aligned}
\int_{\bar{\tau}^{-1}(t)}^{t} \bar{b}(y)dy &\geq \max\{\int_{\tau_1^{-1}(t)}^{t} b_1(y)dy, \int_{\tau_2^{-1}(t)}^{t} b_2(y)dy\} \\
&\geq \min\{\int_{\tau_1^{-1}(t)}^{t} b_1(y)dy, \int_{\tau_2^{-1}(t)}^{t} b_2(y)dy\} \\
&\geq \int_{\underline{\tau}^{-1}(t)}^{t} \underline{b}(y)dy
\end{aligned} \tag{32}$$

and, therefore, for $t \in [t(l), t(l+1)]$,

$$\begin{aligned}
|v_1(t) - v_2(t)| &= |\int_{\tau_1^{-1}(t)}^{t} b_1(y)dy - \int_{\tau_2^{-1}(t)}^{t} b_2(y)dy| \\
&\leq \int_{\bar{\tau}^{-1}(t)}^{t} \bar{b}(y)dy - \int_{\underline{\tau}^{-1}(t)}^{t} \underline{b}(y)dy \\
&= \int_{\bar{\tau}^{-1}(t)}^{\underline{\tau}^{-1}(t)} \bar{b}(y)dy + \int_{\underline{\tau}^{-1}(t)}^{T} (\bar{b}(y) - \underline{b}(y))dy \\
&\leq B_a(\underline{\tau}^{-1}(t) - \bar{\tau}^{-1}(t)) + \int_{0}^{T_a} (\bar{b}(y) - \underline{b}(y))dy \\
&\leq B_a(\underline{\tau}^{-1}(t) - \bar{\tau}^{-1}(t)) + T_a^{\frac{1}{2}} ||\bar{b} - \underline{b}|| \\
&\leq T_a^{\frac{1}{2}} ||b_1 - b_2|| \sum_{n=0}^{l} (B_a L_\zeta L_s)^n
\end{aligned} \tag{33}$$

where the last inequality of (33) yields from the following estimation, which is established by the inductive assumption (29) and (30) for $i \leq l$ and Lemma 7.2,

$$\begin{aligned}
\underline{\tau}^{-1}(t) - \bar{\tau}^{-1}(t) &= \bar{\tau}^{-1}(\bar{\tau}(\underline{\tau}^{-1}(t))) - \bar{\tau}^{-1}(t) \\
&\leq L_\zeta(\bar{\tau}(\underline{\tau}^{-1}(t)) - t) \\
&= L_\zeta(\bar{\tau}(\underline{\tau}^{-1}(t)) - \underline{\tau}(\underline{\tau}^{-1}(t)))
\end{aligned}$$

$$
\begin{aligned}
&= L_\zeta(s_a(\bar{v}(\underline{\tau}^{-1}(t)) - s_a(\underline{v}(\underline{\tau}^{-1}(t))))\\
&\leq L_\zeta L_s(\bar{v}(\underline{\tau}^{-1}(t))) - \underline{v}(\underline{\tau}^{-1}(t))))\\
&\leq T_a^{\frac{1}{2}} L_\zeta L_s ||b_1 - b_2|| \sum_{n=0}^{l-1} (B_a L_\zeta L_s)^n. \qquad (34)
\end{aligned}
$$

Utilizing (33), we have, for $t \in [t(l), t(l+1)]$,

$$
\begin{aligned}
|\tau_1(t) - \tau_2(t)| &= |s_a(v_1(t)) - s_a(v_2(t))|\\
&\leq L_s |v_1(t) - v_2(t)|\\
&\leq L_s T_a^{\frac{1}{2}} ||b_1 - b_2|| \sum_{n=0}^{l} (B_a L_\zeta L_s)^n.
\end{aligned}
$$

Therefore, by induction, we have shown that (29) and (30) are valid for all $i \geq 1$. Since $t(i+1) - t(i) = \underline{\tau}(t(i)) - t(i) = s_a(\underline{v}(t(i))) \geq s_a(0) > 0$, there exists an N such that $t(N) > T_a$. Therefore, (29) and (30) hold true for all $t \in [0, T_a]$, with N in place of i at the right-hand sides. Namely, for all $t \in [0, T_a]$,

$$|v_1(t) - v_2(t)| \leq \bar{L}_v ||b_1 - b_2||$$

and

$$|\tau_1(t) - \tau_2(t)| \leq \bar{L}_\tau ||b_1 - b_2||$$

where we denote $T_a^{\frac{1}{2}} \sum_{n=0}^{N-1} (B_a L_\zeta L_s)^n$ by $\bar{L}_v$, and denote $T_a^{\frac{1}{2}} L_\zeta \sum_{n=0}^{N-1} (B_a L_\zeta L_s)^n$ by $\bar{L}_\tau$.

Our desired results now follow immediately, by letting $L_v = T^{\frac{1}{2}} \bar{L}_v$ and $L_\tau = T^{\frac{1}{2}} \bar{L}_\tau$,

$$||\mathbf{v_a}(b_1) - \mathbf{v_a}(b_2)|| = ||v_1 - v_2|| \leq T^{\frac{1}{2}} \max_{0 \leq t \leq T} \{|v_1(t) - v_2(t)|\} \leq T^{\frac{1}{2}} \bar{L}_v ||b_1 - b_2|| = L_v ||b_1 - b_2||,$$

and

$$||\mathbf{s_a}(\mathbf{b_1}) - \mathbf{s_a}(\mathbf{b_2})|| = ||\tau_1 - \tau_2|| \leq T^{\frac{1}{2}} \max_{0 \leq t \leq T} \{|\tau_1(t) - \tau_2(t)|\} \leq T^{\frac{1}{2}} \bar{L}_\tau ||b_1 - b_2|| = L_\tau ||b_1 - b_2||.$$

That is, the operators $\mathbf{v_a}$ and $\mathbf{s_a}$ depend Lipschitz continuously on $b_a \in D_a$, for any two arbitrary elements b_1 and b_2. This completes the proof. □

8 Appendix C: Proof of Theorem 3.2

Proof of Theorem 3.2: Let $b_1, b_2 \in D_a$, if $\mathbf{v_a}(b_1) = \mathbf{v_a}(b_2)$, then $v_1(t) = v_2(t), \forall t \in [0, \tau_1(T_a)]$. It follows immediately that

$$\tau_1'(t) = \tau_2'(t),$$

for almost all t in $[0, T_a]$. Therefore, we have

$$\int_0^t (b_1(x) - b_2(x))dx = 0, \quad \forall t \in [0, s(0)]$$

and

$$\int_{\tau_1^{-1}(t)}^t (b_1(x) - b_2(x))dx = 0, \forall t \in [s(0), \tau_1(T_a)].$$

Thus we have for all $t \in [0, \tau_1(T_a)]$

$$\int_0^t (b_1(x) - b_2(x))dx = 0.$$

It turns out that $b_1(t) = b_2(t)$ for almost all $t \in [0, \tau_1(T_a)]$. The proof is complete. □

9 Appendix D: Proof of Theorem 3.3

To prove the theorem, we need the following lemmas.

Lemma 9.1 *Let (D_a, s_a) satisfy the flow/arc cooperation conditions. Let $b_1, b_2 \in D_a$ be any two flow entry rates, and let τ_1, τ_2 be respectively their exit times. For every $\bar{t} \in [0, T_a]$ and $\epsilon > 0$, there exist $\delta_b > 0$ and $\delta_\tau > 0$ such that, when $||\tau_1' - \tau_2'||_{[0,\bar{t}]} < \delta_\tau$ and $||b_1 - b_2|| < \delta_b$,*

$$||(\tau_1^{-1})' - (\tau_2^{-1})'||_{\bar{T}} < \epsilon$$

where $||\cdot||_{\bar{T}}$ refers to the L^2 norm where the integration takes over $\bar{T} = [\tau_i(0), \tau_i(\bar{t})]$, $i = 1, or\ \ 2$, such that $\tau_i(\bar{t}) = \min\{\tau_1(\bar{t}), \tau_2(\bar{t})\}$.

Proof. It is clear that $\tau_1(0) = \tau_2(0) = s_a(0)$. Without loss of generality, we assume that $\tau_1(\bar{t}) \leq \tau_2(\bar{t})$, therefore, $\bar{T} = [\tau_1(0), \tau_1(\bar{t})]$ in the statement of the lemma, and we want to prove that

$$||(\tau_1^{-1})' - (\tau_2^{-1})'||_{[\tau_1(0), \tau_1(\bar{t})]} < \epsilon$$

under the conditions of the lemma. Since

$$\begin{aligned} |(\tau_1^{-1})'(t) - (\tau_2^{-1})'(t)| &= \frac{|\tau_1'(\tau_1^{-1}(t)) - \tau_2'(\tau_2^{-1}(t))|}{\tau_1'(\tau_1^{-1}(t))\tau_2'(\tau_2^{-1}(t))} \\ &\leq \frac{|\tau_1'(\tau_1^{-1}(t)) - \tau_1'(\tau_2^{-1}(t))| + |\tau_1'(\tau_2^{-1}(t)) - \tau_2'(\tau_2^{-1}(t))|}{\tau_1'(\tau_1^{-1}(t))\tau_2'(\tau_2^{-1}(t))}, \end{aligned}$$

we have, using the Minkovski inequality,

$$\begin{aligned} \|(\tau_1^{-1})' - (\tau_2^{-1})'\|_{\mathcal{T}} &= \left(\int_{\mathcal{T}} |(\tau^{-1})'(t) - (\tau^{-1})'(t)|^2 dt\right)^{1/2} \\ &\leq \left(\int_{\mathcal{T}} \frac{|\tau_1'(\tau_1^{-1}(t) - \tau_1'(\tau_2^{-1}(t)|^2}{(\tau_1'(\tau_1^{-1}(t)\tau_2'(\tau_2^{-1}(t))^2} dt\right)^{1/2} \\ &\quad + \left(\int_{\mathcal{T}} \frac{|\tau_1'(\tau_2^{-1}(t) - \tau_2'(\tau_2^{-1}(t)|^2}{(\tau_1'(\tau_1^{-1}(t)\tau_2'(\tau_2^{-1}(t))^2} dt\right)^{1/2} \\ &\leq (I)^{1/2} + (II)^{1/2}. \end{aligned} \tag{35}$$

We now proceed to estimate the right-hand side of inequality (35). We first estimate for

$$I = \int_{\mathcal{T}} \frac{|\tau_1'(\tau_1^{-1}(t) - \tau_1'(\tau_2^{-1}(t)|^2}{(\tau_1'(\tau_1^{-1}(t)\tau_2'(\tau_2^{-1}(t))^2} dt.$$

According to Lusin's Theorem of real analysis (Rudin, 1987), for any $\epsilon_1 > 0$, there exists a continuous function $\gamma_1 = \gamma_1(\epsilon_1)$ such that $m(\{\tau_1' \neq \gamma_1\}) < \epsilon_1$ and $\max|\gamma_1(t)| \leq \sup|\tau_1'(t)|$. Further more, for any $\epsilon_2 > 0$, we can find a polynomial function γ_2, with Lipschitz constant L_2 on $[0, T_a]$ such that

$$|\gamma_1(t) - \gamma_2(t)| \leq \epsilon_2, \quad \forall t \in [0, T_a].$$

We note that $|\tau'|$ has an up bound over its domain, denoted by $B_{\tau'}$, which may come from the following estimation:

$$\begin{aligned} |\tau'(t)| &= |1 + s_a'(v_a(t))v_a'(t)| \\ &= 1 + |s_a'(v_a))(b_a(t) - e_a(t))| \\ &\leq 1 + |s_a(v_a(t))||b_a(t) - b_a(\tau_a^{-1}(t))/\tau_a'(\tau_a^{-1}(t))| \\ &\leq 1 + B_s B_b(1 + \zeta)/\zeta \end{aligned}$$

where B_s is the bound of $|s_a'|$, B_b is the bound of $\|b_a(t)\|$, and ζ is the lower bound of τ' over their respective domains.

Following the above definitions and notations, we first have:

$$|\tau_1'(\tau_1^{-1}(t)) - \tau_1'(\tau_2^{-1}(t))|^2 \leq$$

$$|\tau_1'(\tau_1^{-1}(t)) - \gamma_1(\tau_1^{-1}(t)) - \tau_1'(\tau_2^{-1}(t) + \gamma_1(\tau_2^{-1}(t))|^2$$

$$+|\gamma_1(\tau_1^{-1}(t)) - \gamma_2(\tau_1^{-1}(t)) - \gamma_1(\tau_2^{-1}(t)) + \gamma_2(\tau_2^{-1}(t))|^2$$

$$+|\gamma_2(\tau_1^{-1}(t)) - \gamma_2(\tau_2^{-1}(t))|^2. \tag{36}$$

The first item at the right-hand side of (36) is zero except on a subset of $[0, T_a]$ with Lebesgue measure ϵ_1, where it is bounded by $4B_{\tau_1'}$. The second item is bounded by $2\epsilon_2$. And the third item is bounded by

$$L_2^2|\tau_1^{-1}(t) - \tau_2^{-1}(t)|^2 \leq L_2^2 L_\zeta^2 |\tau_2(\tau_1^{-1}(t)) - \tau_1(\tau_1^{-1}(t))|^2 \tag{37}$$

where L_ζ is the Lipschitz constant of τ_2^{-1} corresponding to ζ as in Lemma 7.2. Consequently, the estimation for I follows:

$$I \leq \int_{T \cap (\tau_1' \neq \gamma_1)} \frac{|\tau_1'(\tau_1 - 1(t)) - \tau_1'(\tau_2^{-1}(t))|^2}{\tau_1'(\tau_1^{-1}(t))\tau_2'(\tau_2^{-1}(t))|^2} dt$$

$$+ \int_{T \cap (\tau_1' \neq \gamma_1)} \frac{|\gamma_1(\tau_1 - 1(t)) - \gamma_2(\tau_1 - 1(t)) - \gamma_1(\tau_2 - 1(t)) + \gamma_2(\tau_2 - 1(t))|^2}{\tau_1'(\tau_1^{-1}(t))\tau_2'(\tau_2^{-1}(t))|^2} dt$$

$$+ \int_{T \cap (\tau_1' \neq \gamma_1)} \frac{|\gamma_2(\tau_1 - 1(t)) - \gamma_2(\tau_2 - 1(t))|^2}{\tau_1'(\tau_1^{-1}(t))\tau_2'(\tau_2^{-1}(t))|^2} dt$$

$$\leq \frac{\epsilon_1 B_{\tau_1'}^2}{\zeta_1^2 \zeta_2^2} + \frac{4\epsilon_2^2}{\zeta_1^2 \zeta_2^2}(\tau_1(\bar{t}) - \tau_1(0)) + \frac{L_2^2 L_{\zeta_2}^2}{\zeta_1 \zeta_2^2} \int_{\tau_1(0)}^{\tau_1(T_a)} \frac{|\tau_2(\tau_1^{-1}(t)) - \tau_1(\tau_1^{-1}(t))|^2}{|\tau_1'(\tau_1^{-1}(t))|} dt. \tag{38}$$

Since ϵ_1 can be arbitrarily small for given τ_1, τ_2, we can choose ϵ_1 sufficiently small so that

$$\frac{\epsilon_1 B_{\tau_1'}^2}{\zeta_1^2 \zeta_2^2} < \frac{\epsilon^2}{8}. \tag{39}$$

Since ϵ_2 can be arbitrarily small for given ϵ_1 and γ_1, so we can choose ϵ_2 small enough so that

$$\frac{4\epsilon_2^2}{\zeta_1^2 \zeta_2^2}(\tau_1(\bar{t}) - \tau_1(0)) < \frac{\epsilon^2}{8}. \tag{40}$$

By letting $\mu = \tau_1^{-1}(t)$, we have

$$\frac{L_2^2 L_{\zeta_2}^2}{\zeta_1 \zeta_2^2} \int_{\tau_1(0)}^{\tau_1(T_a)} \frac{|\tau_2(\tau_1^{-1}(t)) - \tau_1(\tau_1^{-1}(t))|^2}{|\tau_1'(\tau_1^{-1}(t))|} dt$$

$$= \frac{L_2^2 L_{\zeta_2}^2}{\zeta_1 \zeta_2^2} \int_0^{T_a} |\tau_2(u) - \tau_1(u)|^2 du$$

$$= \frac{L_2^2 L_{\zeta_2}^2}{\zeta_1 \zeta_2^2} ||\tau_1 - \tau_2||_{[0,T_a]}^2. \tag{41}$$

Substituting (39), (40) and (41) into (38), we obtain

$$\begin{aligned} I &\leq \frac{\epsilon^2}{8} + \frac{\epsilon^2}{8} + \frac{L_2^2 L_{\zeta_2}^2}{\zeta_1 \zeta_2^2} ||\tau_1 - \tau_2||_{[0,T_a]}^2 \\ &\leq \frac{\epsilon^2}{4} + \frac{L_2^2 L_{\zeta_2}^2}{\zeta_1 \zeta_2^2} L_\tau^2 ||b_1 - b_2||_{[0,T_a]}^2 \end{aligned} \tag{42}$$

where L_τ is the Lipschitz constant of operator $\mathbf{s_a}$ with respect to $\mathbf{b}$ as in Theorem 3.1.

On the other hand, with notice to $\tau_1(\bar{t}) \leq \tau_2(\bar{t})$, we have,

$$\begin{aligned} II &= \int_{\tau_2(0)}^{\tau_2(\bar{t})} \frac{|\tau_1'(\tau_2^{-1}(t)) - \tau_2'(\tau_2^{-1}(t))|^2}{\tau_1'(\tau_1^{-1}(t))\tau_2'(\tau_2^{-1}(t))|^2} dt \\ &\leq \int_{\tau_2(0)}^{\tau_2(\bar{t})} \frac{|\tau_1'(\tau_2^{-1}(t)) - \tau_2'(\tau_2^{-1}(t))|^2}{\zeta_1^2 \zeta_2 \tau_2'(\tau_2^{-1}(t))} dt \\ &= \frac{1}{\zeta_1^2 \zeta_2} \int_0^{\bar{t}} |\tau_1'(u) - \tau_2'(u)|^2 du \\ &= \frac{1}{\zeta_1^2 \zeta_2} ||\tau_1' - \tau_2'||_{[0,\bar{t}]}^2. \end{aligned} \tag{43}$$

Therefore, we are able to conclude that:

$$||(\tau_1^{-1})' - (\tau_2^{-1})'||_{[\tau_1(0),\tau_2(\bar{t})]}$$

$$\leq \left(\frac{\epsilon^2}{4} + \frac{L_2^2 L_{\zeta_2}^2}{\zeta_1 \zeta_2^2} L_\tau^2 ||b_1 - b_2||_{[0,T_a]}^2 \right)^{\frac{1}{2}} + \left(\frac{1}{\zeta_1^2 \zeta_2} ||\tau_1' - \tau_2'||_{[0,\bar{t}]}^2 \right)^{\frac{1}{2}}$$

$$\leq \frac{\epsilon}{2} + \frac{L_2 L_{\zeta_2} L_\tau}{\sqrt{\zeta_1 \zeta_2}} ||b_2 - b_1|| + \frac{1}{\zeta_1 \sqrt{\zeta_2}} ||\tau_1' - \tau_2'||_{[0,\bar{t}]}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon$$

where the last inequality may be realized by choosing $\delta_b = (\sqrt{\zeta_1}\zeta_2\epsilon)/(4L_2L_{\zeta_2}L_\tau)$ and $\delta_\tau = (\zeta_1\sqrt{\zeta_2}\epsilon)/4$. So, the proof of the lemma is complete. □

Lemma 9.2 *Let D_a and s_a satisfy the flow/arc cooperation conditions. For every $\bar{t} \in [0, T_a]$ and $\epsilon > 0$, there exist $\delta_b > 0$ and $\delta_\tau > 0$ such that, when $||\tau_1' - \tau_2'||_{[0,\bar{t}]} < \delta_\tau$ and $||b_1 - b_2|| < \delta_b$,*

$$||e_1 - e_2||_{\bar{T}} < \epsilon$$

where $||\cdot||_{\bar{T}}$ refers to the L^2 norm where the integration takes over $\bar{T} = [\tau_i(0), \tau_i(\bar{t})]$, $i = 1$, or 2, such that $\tau_i(\bar{t}) = \min\{\tau_1(\bar{t}), \tau_2(\bar{t})\}$.

Proof.

With notice that $e_1(t) = e_2(t) = 0, \forall t < s_a(0)$, we need only to pay attention to $t \geq s_a(0)$. Also, one always has $\tau_1(0) = \tau_2(0) = s_a(0)$. Let us assume that, without the loss of generality, $\tau_1(\bar{t}) \leq \tau_2(\bar{t})$, therefore, $\bar{T} = [\tau_1(0), \tau_1(\bar{t})]$ in the statement of the lemma, and we want to prove that

$$||e_1 - e_2||_{[\tau_1(0),\tau_1(\bar{t})]} < \epsilon.$$

By definition, it follows from Minkowski's Inequality that:

$$\begin{aligned}
||e_1 - e_2||^2_{[\tau_1(0),\tau_1(\bar{t})]} &= \int_{\tau_1(0)}^{\tau_1(\bar{t})} |e_1(t) - e_2(t)|^2 dt \\
&= \int_{\tau_1(0)}^{\tau_1(\bar{t})} |b_1(\tau_1^{-1}(t))(\tau_1^{-1})'(t) - b_2(\tau_2^{-1}(t))(\tau_2^{-1})'(t)|^2 dt \\
&\leq \int_{\tau_1(0)}^{\tau_1(\bar{t})} |b_1(\tau_1^{-1}(t))(\tau_1^{-1})'(t) - b_2(\tau_1^{-1}(t))(\tau_1^{-1})'(t)|^2 dt \\
&\quad + \int_{\tau_1(0)}^{\tau_1(\bar{t})} |b_2(\tau_1^{-1}(t))(\tau_1^{-1})'(t) - b_2(\tau_2^{-1}(t))(\tau_1^{-1})'(t)|^2 dt \\
&\quad + \int_{\tau_1(0)}^{\tau_1(\bar{t})} |b_2(\tau_2^{-1}(t))(\tau_1^{-1})'(t) - b_2(\tau_2^{-1}(t))(\tau_2^{-1})'(t)|^2 dt \\
&= (I) + (II) + (III)
\end{aligned}$$

where $(I), (II), (III)$ indicate the first, second, and third items at the right-hand side of the above inequality. In the following, we will estimate each of these three items. First, we have

$$(I) = \int_{\tau_1(0)}^{\tau_1(\bar{t})} |b_1(\tau_1^{-1}(t))(\tau_1^{-1})'(t) - b_2(\tau_1^{-1}(t))(\tau_1^{-1})'(t)|^2 dt$$

$$= \int_0^{\bar{t}} |b_1(u) - b_2(u)|^2 du$$
$$= ||b_1 - b_2||^2_{[0,\bar{t}]}. \tag{44}$$

Secondly, we realize the fact that

$$|\tau_1^{-1}(t) - \tau_2^{-1}(t)| = |\tau_2^{-1}(\tau_2(\tau_1^{-1}(t))) - \tau_2^{-1}(\tau_2(\tau_2^{-1}(t)))|$$

$$\leq L_{\zeta_2}|\tau_2(\tau_1^{-1}(t)) - t| = L_{\zeta_2}|\tau_2(\tau_1^{-1}(t)) - \tau_1(\tau_1^{-1}(t))|$$

where L_{ζ_2} is the Lipschitz constant of τ_2^{-1} corresponding to ζ as in Lemma 3.2. This fact is utilized in the following estimation:

$$\begin{aligned}
(II) &= \int_{\tau_1(0)}^{\tau_1(\bar{t})} |b_2(\tau_1^{-1}(t))(\tau_1^{-1})'(t) - b_2(\tau_2^{-1}(t))(\tau_1^{-1})'(t)|^2 dt \\
&\leq \frac{1}{\zeta_1} \int_{\tau_1(0)}^{\tau_1(\bar{t})} |b_2(\tau_1^{-1}(t)) - b_2(\tau_2^{-1}(t))|^2 (\tau_1^{-1})'(t) dt \\
&\leq \frac{L_{b_2}^2}{\zeta_1} \int_{\tau_1(0)}^{\tau_1(\bar{t})} \|\tau_1^{-1}(t) - \tau_2^{-1}(t)\|^2 (\tau_1^{-1})'(t) dt \\
&\leq \frac{L_{b_2}^2}{\zeta_1} \int_{\tau_1(0)}^{\tau_1(\bar{t})} L_{\zeta_2}^2 |\tau_2(\tau_1^{-1}(t)) - \tau_1(\tau_1^{-1}(t))|^2 (\tau_1^{-1})'(t) dt \\
&= \frac{L_{b_2}^2 L_{\zeta_2}^2}{\zeta_1} \int_0^{\bar{t}} |\tau_2(u) - \tau_1(u)|^2 du \qquad (45) \\
&= \frac{L_{b_2}^2 L_{\zeta_2}^2}{\zeta_1} ||\tau_2 - \tau_1||^2_{[0,\bar{t}]} \\
&\leq \frac{L_{b_2}^2 L_{\zeta_2}^2 L_\tau^2}{\zeta_1} ||b_2 - b_1||^2_{[0,\bar{t}]} \qquad (46)
\end{aligned}$$

where L_{b_2} is the Lipschitz constant for b_2, L_τ is the Lipschitz constant for τ and the last inequality follows from Theorem 3.1.

Finally, we have the following estimation

$$\begin{aligned}
(III) &= \int_{\tau_1(0)}^{\tau_1(\bar{t})} |b_2(\tau_2^{-1}(t))(\tau_1^{-1})'(t) - b_2(\tau_2^{-1}(t))(\tau_2^{-1})'(t)|^2 dt \\
&\leq B_{b_2}^2 \int_{\tau_1(0)}^{\tau_1(\bar{t})} |(\tau_1^{-1})'(t) - (\tau_2^{-1})'(t)|^2 dt \\
&= B_{b_2}^2 ||(\tau_1^{-1})' - (\tau_2^{-1})'||^2_{[\tau_1(0),\tau_1(\bar{t})]}.
\end{aligned}$$

According to Lemma 9.1, for any $\epsilon > 0$, there exist a $\delta_b > 0$ and a $\delta_\tau > 0$, such that

$$||(\tau_1^{-1})' - (\tau_2^{-1})'||^2_{[\tau_1(0),\tau_1(\bar{t})]} < \frac{\epsilon}{3B_{b_2}}$$

when $||b_1 - b_2|| < \delta_b, ||\tau_1' - \tau_2'|| < \delta_\tau$.

Therefore, under the above conditions, one has

$$(III) \leq \frac{\epsilon^2}{9}. \tag{47}$$

By choosing $||b_1 - b_2|| < \min\{\frac{\epsilon}{3}, \delta_b, \frac{\sqrt{\zeta_1}\epsilon}{3L_{b_2}L_{\zeta_2}L_\tau}\}, ||\tau_1' - \tau_2'|| < \delta_\tau$, and following our estimations (44), (46), and (47), one can conclude with $(I)^{\frac{1}{2}} < \frac{\epsilon}{3}, (II)^{\frac{1}{2}} < \frac{\epsilon}{3}, (III)^{\frac{1}{2}} < \frac{\epsilon}{3}$, and therefore $||e_1 - e_2||_\mathcal{T} < \epsilon$. The proof is complete. □

Lemma 9.3 *Let D_a and s_a satisfy the flow/arc coopreration condition. Let $b_1, b_2 \in D_a$. Then if $||e_1 - e_2||_{t \leq t''}$ and $||b_1 - b_2||_{t \leq t''}$ are sufficiently small, then $||\tau_1' - \tau_2'||_{t \leq t''}$ is also sufficiently small.*

Proof. The assertion follows from the following:

$$\begin{aligned} |\tau_1'(t) - \tau_2'(t)| &< \frac{1}{B}|b_1(t) - b_2(t) + e_2(t) - e_1(t)| \\ &\leq \frac{1}{B}|b_1(t) - b_2(t)| + \frac{1}{B}|e_1(t) - e_2(t)|. \end{aligned}$$

□

Proof of Theorem 3.3: We only need to show that for any $b_1, b_2 \in D_a$, $||\tau_1' - \tau_2'||_{t \leq s_a(0)}$ is sufficiently small when $||b_1 - b_2||$ is sufficiently small. Then the theorem will follow Lemma 9.2 and Lemma 9.3.

$$\begin{aligned} ||\tau_1' - \tau_2'||_{t \leq s_a(0)} &= \left(\int_0^{s_a(0)} |\tau_1'(t) - \tau_2'(t)|^2 dt\right)^{1/2} \\ &= \left(\int_0^{s_a(0)} |s_a'(v_1(t))b_1(t) - s_a'(v_2(t))b_2(t)|^2 dt\right)^{1/2} \\ &< \frac{1}{B}\left(\int_0^{s_a(0)} |b_1(t) - b_2(t)|^2 dt\right)^{1/2} \leq \frac{1}{B}||b_1 - b_2||. \end{aligned}$$

This completes the proof. □

10 Appendix E: Proof of Theorem 4.1

First we prove the following lemma.

Lemma 10.1 *Let (N, A) be a road network with arc traversal functions $s_a, a \in A$ and $h = (h_k, k \in K)$ be a set of path departure rate distributions for the network. Assume that h and s satisfy the flow/network cooperation conditions. If for any arc $a \in A$, $b_a|_{t \le T_i}$ continuously depends on h for $i \ge 1$, then $b_a|_{t \le T_{i+1}}$ also continuously depends on h.*

Proof. We show that, under the given condition, for an arbitrary $\epsilon > 0$, there exists a $\delta(\epsilon)$ such that for any $a \in A$ it holds

$$||b_a^1 - b_a^2||_{t \le T_{i+1}} < \epsilon$$

when $||h_1 - h_2|| < \delta$.

As before, we use index (ka) to express arc a on path k, $(ka-)$ the previous arc of arc a on path k. We set a virtual arc a_{k0} for each path$k$ and assign a_{k0} as the previous arc of the first (real) arc of path k. Accordingly, $h_k(t)$ is the exit flow rate function of the virtual arc a_{k0}.

By (7), $b_a(t) = \sum_{k \in P_a} e_{ka-}(t)$. It follows that

$$||b_a^1 - b_a^2||_{t \le T_{i+1}} \le \sum_{k \in P_a} ||e_{ka-}^1 - e_{ka-}^2||_{t \le T_{i+1}}.$$

It follows that if for all $a \in A$, it holds

$$||e_{ka-}^1 - e_{ka-}^2||_{t \le T_{i+1}} < \epsilon_1 = \frac{1}{|N| - 1}\epsilon,$$

where $|N|$ is the cardinal number of the set N. The lemma is proved.

By Theorem 3.3, we have that for given $\epsilon_1 = \frac{1}{|N|-1}\epsilon$, there exists $\delta_1(\epsilon_1) > 0$ such that

$$||e_{ka-}^1 - e_{ka-}^2||_{t \le T_{i+1}} < \epsilon_1 = \frac{1}{|N| - 1}\epsilon,$$

when $||b_a^1 - b_a^2||_{t \le T_i} < \delta_1$.

Because $b_a|_{t \le T_i}$ continuously depends on h, for any two $h_1, h_2 \in D$ and corresponding arc flow entry rate b_a^1, b_a^2, it holds for an arbitrary $\delta_1 > 0$, there exists a $\delta(\delta_1) > 0$ such that

$$||b_a^1 - b_a^2||_{t \le T_i} < \delta_1,$$

when $||h_1 - h_2|| < \delta$. This completes the proof. □

Proof of Theorem 4.1: Let T_* be the time such that for $t > T_*$ the system is inactive. T_* is a finite real number. Then there exists a positive number n such that $T_n = n\alpha \geq T_*$. We need to show that $b_a|_{t \leq T_n}$ is continuously depends on h. To do this, we only need to show $b_a|_{t \leq T_1}$ is continuously depends on h by Lemma 10.1.

For $i = 1$, it holds that $b_a(t) = h_k(t), t \in [0, T_1]$ if a is the first arc of the path k, otherwise $b_a(t) = 0, t \in [0, T_1]$. Clearly it holds that for any $a \in A$

$$||b_a^1 - b_a^2||_{t \leq T_1} = ||h_k^1 - h_k^2||_{t \leq T_1} \leq ||h_1 - h_2||.$$

Then for an arbitrary $\epsilon_0 > 0$, we may take $\delta = \epsilon_0$ such that $||b_a^1 - b_a^2||_{t \leq T_1} < \epsilon_0$ when $||h_1 - h_2|| < \delta$. This completes the proof. □

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